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Infrared Regularization of Superstring Theory and the One-Loop Calculation of Coupling Constants

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ABSTRACT

Infrared regularized versions of 4-D N=1 superstring ground states are constructed by curving the spacetime. A similar regularization can be performed in field theory. For the IR regularized string ground states we derive the exact one-loop effective action for non-zero U(1) or chromo-magnetic fields as well as gravitational and axionic-dilatonic fields. This effective action is IR and UV finite. Thus, the one-loop corrections to all couplings (gravitational, gauge and Yukawas) are unambiguously computed. These corrections are necessary for quantitative string superunification predictions at low energies. The one-loop corrections to the couplings are also found to satisfy Infrared Flow Equations.

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1 Introduction

The four-dimensional superstring solutions in a flat background [1]-[7] define, at low energy, effective supergravity theories [8]-[11]. A class of them successfully extends the validity of the standard model up to the string scale, M_{str}. The first main property of superstrings is that they are ultraviolet-finite theories (at least perturbatively). Their second important property is that they unify gravity with all other interactions. This unification does not include only the gauge interactions, but also the Yukawa ones as well as the interactions among the scalars. This String Hyper Unification (SHU) happens at large energy scales $E_t \sim \mathcal{O}(M_{str}) \sim 10^{17}$ GeV. At this energy scale, however, the first excited string states become important and thus the whole effective low energy field theory picture breaks down [12]-[15]. Indeed, the effective field theory of strings is valid only for $E_t \ll M_{str}$ by means of the $\mathcal{O}(E_t/M_{str})^2$ expansion. It is then necessary to evolve the SHU predictions to a lower scale $M_U < M_{str}$ where the effective field theory picture makes sense. Then, at M_U, any string solution provides non-trivial relations between the gauge and Yukawa couplings, which can be written as*

$$\frac{k_i}{\alpha_i(M_U)} = \frac{k_j}{\alpha_j(M_U)} + \Delta_{ij}(M_U). \tag{1.1}$$

The above relation looks very similar to the well-known unification condition in Supersymmetric Grand Unified Theories (SuSy-GUTs) where the unification scale is about $M_U \sim 10^{16}$ GeV and $\Delta_{ij}(M_U) = 0$ in the $\bar{DR}$ renormalization scheme; in SuSy-GUTs the normalization constants k_i are fixed *only* for the gauge couplings ($k_1 = k_2 = k_3 = 1$, $k_{em} = \frac{3}{8}$), but there are no relations among gauge and Yukawa couplings at all. In string effective theories, however, the normalization constants (k_i) are known for both gauge and Yukawa interactions. Furthermore, $\Delta_{ij}(M_U)$ are calculable $finite$ quantities for any particular string solution. Thus, the predictability of a given string solution is extended for all low energy coupling constants $\alpha_i(M_Z)$ once the string-induced corrections $\Delta_{ij}(M_U)$ are determined.

This determination however, requests string computations which we did not know, up to now, how to perform in generality. It turns out that $\Delta_{ij}(M_U)$ are non-trivial functions of the vacuum expectation values of some gauge singlet fields [9, 10, 11], $\langle T_A \rangle = t_A$, the so-called moduli (the moduli fields are flat directions at the string classical level and they remain flat in string perturbation theory, in the exact supersymmetric limit). The $\Delta_{ij}(t_A)$ are target space duality invariant functions, which depend on the particular string solution. Partial results for Δ_{ij} exist [9, 10, 11] in the exact supersymmetric limit in many string solutions based on orbifold [2] and fermionic constructions [5]. As we will see later Δ_{ij} are, in principle, well defined calculable quantities once we perform our calculations at the string level where all interactions including gravity are consistently defined. The full string corrections to the coupling constant unification, $\Delta_{ij}(M_U)$, as well as the string corrections associated to the soft supersymmetry-breaking parameters

*The logarithmic part was calculated for the first time in string theory in [16].

$$m_0,\ m_{1/2},\ A,\ B \text{ and } \mu, \text{ at } M_U,$$

are of main importance, since they fix the strength of the gauge and Yukawa interactions, the full spectrum of the supersymmetric particles as well as the Higgs and the top-quark masses at the low energy range $M_Z \leq E_t \leq \mathcal{O}(1)$ TeV.

In the case where supersymmetry is broken [17, 18] only semi-quantitative results can be obtained at present; a much more detailed study and understanding are necessary which is related to the structure of soft breaking terms after the assumed supersymmetry breaking [19].

The main obstruction in determining the exact form of the string radiative corrections $\Delta_{ij}(M_U)$ is strongly related to the infrared divergences of the $\langle [F^a_{\mu\nu}]^2 \rangle$ two-point correlation function in superstring theory. In field theory, we can avoid this problem using off-shell calculations. In first quantized string theory we cannot do that since we do not know how to go off-shell. Even in field theory there are problems in defining an infrared regulator for chiral fermions especially in the presence of spacetime supersymmetry.

In [20] it was suggested to use a specific spacetime with negative curvature in order to achieve consistent regularization in the infrared. The proposed curved space however is not useful for string applications since it does not correspond to an exact super-string solution.

Recently, exact and stable superstring solutions have been constructed using special four-dimensional spaces as superconformal building blocks with $\hat{c} = 4$ and $N = 4$ superconformal symmetry [12, 14]. The full spectrum of string excitations for the superstring solutions based on those four-dimensional subspaces, can be derived using the techniques developed in [14]. The main characteristic property of these solutions is the existence of a mass gap, which is proportional to the curvature of the non-trivial four-dimensional spacetime. Comparing the spectrum in a flat background with that in curved space we observe a shifting of all massless states by an amount proportional to the spacetime curvature, $\Delta m^2 = Q^2/4 = \mu^2/2$, where Q is the Liouville background charge and μ is the IR cutoff. What is also interesting is that the shifted spectrum in curved space is equal for bosons and fermions due to the existence of a new space-time supersymmetry defined in curved spacetime [12, 14]. Therefore, our curved spacetime infrared regularization is consistent with supersymmetry and can be used either in field theory or string theory.

In section 2 we define the four-dimensional superconformal system responsible for the IR cutoff and give the modular-invariant partition function for some symmetric orbifold ground states of the string. In section 3 we show how we can deform the theory consistently, by switching on a non-zero gauge field strength background $\langle F^a_{\mu\nu} F^{\mu\nu}_a \rangle = F^2$ or a gravitational one, $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle = \mathcal{R}^2$ and obtain the *exact* regularized partition function $Z(\mu, F, \mathcal{R})$. Our method of constructing this effective action automatically takes into account the back-reaction of the other background fields; stated otherwise, the perturbation that turns on the constant gauge field strength or curvature background is an exact (1,1) integrable perturbation. The second derivative with respect to F of our deformed partition function $\partial^2 Z(\mu, F, \mathcal{R})/\partial F^2$ for $F, \mathcal{R} = 0$ defines without any infrared ambiguities the

complete string one-loop corrections to the gauge coupling constants. In the $\mu \to 0$ limit we recover the known partial results [9, 10, 11]. A preliminary version of our results has appeared in [21].

2 Regulating the Infrared

Any 4-D string solution that can be used to describe particle physics is composed from a 4-D flat spacetime CFT (with $c = (6, 4)$) which provides the universal degrees of freedom (graviton, antisymmetric tensor and dilaton) and some internal CFT (with $c = (9, 22)$) which provides the various particle degrees of freedom (gauge fields, fermions, scalars).

We would like to regularize the IR by turning on background fields associated to the universal degrees of freedom ($G_{\mu\nu}$, $B_{\mu\nu}$, Φ) so that it can be used for 4-D string ground states with arbitrary particle content. This will be done by replacing the 4-D flat spacetime CFT with another CFT which however has to satisfy the following constraints:

1. The string spectrum must have a mass gap μ^2. In particular, chiral fermions should be regulated consistently.

2. We should be able to take the limit $\mu^2 \to 0$.

3. It should have $c = (6, 4)$ so that it can be coupled to any internal CFT with $c = (9, 22)$.

4. It should preserve as many spacetime supersymmetries of the original theory, as possible.

5. We should be able to calculate the regulated quantities relevant for the effective field theory.

6. Vertices for spacetime fields (like $F^a_{\mu\nu}$) should be well defined operators on the world-sheet.

7. The theory should be modular invariant (which guarantees the absence of anomalies).

8. Such a regularization should be possible also at the effective field theory level. In this way, calculations in the fundamental theory can be matched without any ambiguity to those of the effective field theory.

Requirements **3** and **4** imply that the 4-D CFT should have $N = 4$ superconformal symmetry*. If we need to regulate an N=1 spacetime supersymmetric ground state the N=4 requirement can be dropped and N=2 is sufficient. In this case one can use a 4-D CFT with $c = (6 + \epsilon, 4 + \epsilon)$ and an internal CFT with $c = (6 - \epsilon, 22 - \epsilon)$. This would come close to the dimensional regularization of IR divergences used in field theory. However we have good indications that in the limit $\epsilon \to 0$ the internal theory decompactifies so we will not consider this possibility further.

*It is possible to have higher superconformal symmetry but we know of no example that regulates the IR.

There are many N=4 CFTs [22] that can regulate the IR but if we insist on requirement **5**, then we obtain the following list of candidates [25, 12]:

$$\begin{aligned}
\textbf{I}. \quad & W_k^{(4)} && \equiv U(1)_Q \otimes SU(2)_{k_1} \\
\textbf{II}. \quad & C_k^{(4)} && \equiv [SU(2)/U(1)]_k \otimes U(1)_R \otimes U(1)_Q \\
\textbf{III}. \quad & \Delta_k^{(4)}(A) && \equiv [SU(2)/U(1)]_k \otimes [SL(2,R)/U(1)_A]_{k+4} \\
\textbf{IV}. \quad & \Delta_k^{(4)}(V) && \equiv [SU(2)/U(1)]_k \otimes [SL(2,R)/U(1)_V]_{k+4}
\end{aligned}$$

and their N=4 preserving continuous deformations. The CFTs above (and their supersymmetric deformations) are constructed out of conformal subsystems whose characters are known [23, 24] The N=4 superconformal symmetry plays an important role since it indicates the appropriate modular invariant combinations of characters for these systems [12, 14].

In this work we will use system **I** but similar considerations can be advanced for the other systems.

The background charge Q in cases **I** and **II** is related to the level k due to the $N = 4$ algebra, $Q = \sqrt{2/(k+2)}$ and guarantees that $\hat{c} = 4$ for any value of k.

In the limit of weak curvature (large k) the $W_k^{(4)}$ space can be interpreted as a topologically non-trivial four-dimensional manifold of the form $\mathbb{R} \otimes S^3$. The underlying superconformal field theory associated to $W_k^{(4)}$ includes a supersymmetric $SU(2)_k$ WZW model describing the three coordinates of S^3 as well as a non-compact dimension with a background charge, describing the scale factor of the sphere [12, 14]. Furthermore this space admits two covariantly constant spinors and, therefore, respects up to two space-time supersymmetries (in the heterotic case) consistently with the $N = 4$ world-sheet symmetry [26, 12, 14]. The explicit representation of the desired $N = 4$ algebra is derived in [25] and [12], while the target space interpretation as a four-dimensional semi-wormhole space is given in [26].

The basic rules of construction in curved spacetime are similar to that of the orbifold construction [2], the free 2-d fermionic constructions [5], and the Gepner construction [7] where one combines in a modular-invariant way the world-sheet degrees of freedom in a way consistent with unitarity and spin-statistics of the string spectrum.

Our regulated string ground state is of the form $W_k^{(4)} \otimes K^{(6)}$, where $K^{(6)}$ is any appropriate internal CFT. To be explicit, we choose this CFT to be one of the symmetric orbifold models, used in $(2, 2)$ compactifications, although as it will become obvious, this can be done for any "solvable" internal CFT.

Since the world-sheet fermions of the $W_k^{(4)}$ superconformal system are free and since the $K^{(6)}$ internal theory is the same as in the $\mathbb{R}^4 \otimes K^{(6)}$ 4-D superstring solutions, we can easily obtain the partition function of $W_k^{(4)} \otimes K^{(6)}$, Z^W, for k even, in terms of that of $\mathbb{R}^4 \otimes K^{(6)}$, Z^F:

$$Z^W[\mu, \tau, \bar{\tau}] = [\Gamma(SU(2)_k)(\tau, \bar{\tau})] \; Z^F[\tau, \bar{\tau}], \tag{2.1}$$

where $\Gamma(SU(2)_k)$ is nothing but the contribution to the partition function of the bosonic coordinates X^μ of the curved background $W^{(4)}$ divided by the contribution of the four free coordinates of the 4-D flat space,

$$\Gamma(SU(2)_k) = \frac{1}{2}[(\mathrm{Im}\tau)^{\frac{1}{2}}\eta(\tau)\bar{\eta}(\bar{\tau})]^3 \quad \sum_{a,b=0}^{1} Z^{SU(2)}[{}^a_b]. \tag{2.2}$$

$$Z^{SU(2)}[{}^a_b] = e^{-i\pi kab/2}\sum_{l=0}^{k} e^{i\pi bl}\chi_l(\tau)\bar{\chi}_{l+a(k-2l)}(\bar{\tau}) \tag{2.3}$$

where $\chi_l(\tau)$ are the characters of $SU(2)_k$ (see for example [27]) and the integer l is equal to twice the $SU(2)$ spin $l = 2j$. It is necessary to use this orbifoldized version of $SU(2)_k$ in order to project out negative norm states of the $N = 4$ superconformal representations [14]. Note that this factorized form is valid for any 4-D ground state which has $N \leq 2$ spacetime supersymmetry in flat space. This is due to the fact that the $W^{(4)}$ space has two covariantly constant spinors. If the original flat space background has $N = 4$ spacetime supersymmetry then this is broken to $N = 2$ via the coupling of SU(2) spin with the internal manifold. In such a case eq. (2.1) changes and its new form will be presented elsewhere. Also note that even though the vaccum amplitude (2.1) has a factorized form (before the τ integration) this does not imply that the one-loop corrections to couplings have a similar factorized form. In particular as we will see later on, the one-loop correction to the R^2 coupling as well as the corrections to gauge couplings for non-supersymmetric ground states are not factorized.

To obtain the above formula we have used the continuous series of unitary representations of the Liouville characters [14] which are generated by the lowest-weight operators,

$$e^{\beta X_L} \;; \quad \beta = -\frac{1}{2}Q + ip \;, \tag{2.4}$$

having positive conformal weights $h_p = Q^2/8 + p^2/2$. The fixed imaginary part in the momentum $iQ/2$ of the plane waves is due to the non-trivial dilaton motion.

As a particular example we give below the partition function of the $Z_2 \otimes Z_2$ symmetric orbifolds [2, 5], $W_k^{(4)} \otimes T^{(6)}/(Z^2 \otimes Z^2)$, for type-II and heterotic constructions:

$$Z_{II}^{W}[\mu;\tau,\bar{\tau}] \;=\; \frac{\Gamma(SU(2)_k)}{\mathrm{Im}\tau\;\eta^2\bar{\eta}^2} \times \frac{1}{16}\sum_{\alpha,\beta,\bar{\alpha},\bar{\beta}=0}^{1}\;\sum_{h_1,g_1,h_2,g_2} Z_1[{}^{h_1}_{g_1}]Z_2[{}^{h_2}_{g_2}]Z_3[{}^{-h_1-h_2}_{-g_1-g_2}]\times$$

$$(-)^{\alpha+\beta+\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]}{\eta}\frac{\vartheta[{}^{\alpha+h_1}_{\beta+g_1}]}{\eta}\frac{\vartheta[{}^{\alpha+h_2}_{\beta+g_2}]}{\eta}\frac{\vartheta[{}^{\alpha-h_1-h_2}_{\beta-g_1-g_2}]}{\eta} \times (-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha}\bar{\beta}}\frac{\bar{\vartheta}[{}^{\bar{\alpha}}_{\bar{\beta}}]}{\bar{\eta}}\frac{\bar{\vartheta}[{}^{\bar{\alpha}+h_1}_{\bar{\beta}+g_1}]}{\bar{\eta}}\frac{\bar{\vartheta}[{}^{\bar{\alpha}+h_2}_{\bar{\beta}+g_2}]}{\bar{\eta}}\frac{\bar{\vartheta}[{}^{\bar{\alpha}-h_1-h_2}_{\bar{\beta}-g_1-g_2}]}{\bar{\eta}} \tag{2.5}$$

where $Z_i[{}^{h_i}_{g_i}]$ in (2.5) stands for the partition function of two twisted bosons with twists (h_i, g_i). The untwisted part $Z_i[{}^0_0]$ is equal to the moduli-dependent two-dimensional lattice $\Gamma(2,2)[T_i,U_i]/(\eta\bar{\eta})^2$. The definition of the ϑ-function we use is

$$\vartheta[{}^a_b](v|\tau) = \sum_{n\in Z} e^{i\pi\tau(n+a/2)^2+2i\pi(n+a/2)(v+b/2)} \tag{2.6}$$

In the heterotic case, a modular-invariant partition function can be easily obtained using the heterotic map [6, 7]. It consists in replacing in (2.5) the $O(2)$ characters associated to the right-moving fermionic coordinates $\bar{\Psi}^\mu$, with the characters of either $O(10)\otimes E_8$:

$$(-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha}\bar{\beta}}\ \frac{\bar{\vartheta}[^{\bar{\alpha}}_{\bar{\beta}}]}{\bar{\eta}} \to \frac{\bar{\vartheta}[^{\bar{\alpha}}_{\bar{\beta}}]^5}{\bar{\eta}^5}\ \frac{1}{2}\sum_{\gamma,\delta}\frac{\bar{\vartheta}[^{\gamma}_{\delta}]^8}{\bar{\eta}^8} \tag{2.7}$$

or $O(26)$:

$$(-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha}\bar{\beta}}\ \frac{\bar{\vartheta}[^{\bar{\alpha}}_{\bar{\beta}}]}{\bar{\eta}} \to \frac{\bar{\vartheta}[^{\bar{\alpha}}_{\bar{\beta}}]^{13}}{\bar{\eta}^{13}}. \tag{2.8}$$

Using the map above, the heterotic partition function with $E_8\otimes E_6$ unbroken gauge group is:

$$Z^W_{het}[\mu;\tau,\bar{\tau}] = \frac{\Gamma(SU(2)_k)}{\mathrm{Im}\tau\ \eta^2\bar{\eta}^2}\times\frac{1}{16}\sum_{\alpha,\beta,\bar{\alpha},\bar{\beta}=0}^{1}\ \sum_{h_1,g_1,h_2,g_2} Z_1[^{h_1}_{g_1}]Z_2[^{h_2}_{g_2}]Z_3[^{-h_1-h_2}_{-g_1-g_2}]\times$$

$$(-)^{\alpha+\beta+\alpha\beta}\frac{\vartheta[^{\alpha}_{\beta}]}{\eta}\frac{\vartheta[^{\alpha+h_1}_{\beta+g_1}]}{\eta}\frac{\vartheta[^{\alpha+h_2}_{\beta+g_2}]}{\eta}\frac{\vartheta[^{\alpha-h_1-h_2}_{\beta-g_1-g_2}]}{\eta}\times\frac{1}{2}\sum_{\gamma,\delta}\frac{\bar{\vartheta}[^{\gamma}_{\delta}]^8}{\bar{\eta}^8}\frac{\bar{\vartheta}[^{\bar{\alpha}}_{\bar{\beta}}]^5}{\bar{\eta}^5}\frac{\bar{\vartheta}[^{\bar{\alpha}+h_1}_{\bar{\beta}+g_1}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{\bar{\alpha}+h_2}_{\bar{\beta}+g_2}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{\bar{\alpha}-h_1-h_2}_{\bar{\beta}-g_1-g_2}]}{\bar{\eta}} \tag{2.9}$$

The mass spectrum of bosons and fermions in both the heterotic and type-II constructions is degenerate due to the existence of space-time supersymmetry defined in the $W^{(4)}_k$ background. The heterotic constructions are $N=1$ spacetime supersymmetric while in the type-II construction one obtains $N=2$ supersymmetric solutions.

The boson (or fermion) spectrum is obtained by setting to $+1$ (or to -1) the statistical factor, $(-)^{\alpha+\beta+\bar{\alpha}+\bar{\beta}+\alpha\beta+\bar{\alpha}\bar{\beta}}$, in the type-II construction, while one must set the statistical factor $(-)^{\alpha+\beta+\alpha\beta}{=}{+1}$ (or -1) in the heterotic constructions. In order to derive the lower-mass levels we need the behavior of the bosonic and fermionic part of the partition function in the limit where $\mathrm{Im}\tau$ is large ($\mathrm{Im}\tau\to\infty$). This behavior can be easily derived from (2.9),

$$Z^W(\mu;\tau,\bar{\tau}) \longrightarrow \mathrm{C}[\mathrm{Im}\tau]^{-1}\ e^{-\frac{\mathrm{Im}\tau}{2(k+2)}}. \tag{2.10}$$

The above behavior is universal and does not depend on the choice of $K^{(6)}$ internal $N=(2,2)$ space. Only the multiplicity factor C (positive for bosons and negative for fermions) depends on the different constructions and it is always proportional to the number of the lower-mass level states with mass $\mu^2/2 = 1/[2(k+2)] = Q^2/4$. If we replace the $W^{(4)}_k$ with any one of the other $N=4$ $\hat{c}=4$ spaces, $C^{(4)}_k$, $\Delta^{(4)}_k(A,V)$, we get identical infrared mass shift μ.

As we will see in the next section, the induced mass μ acts as a well-defined infrared regulator for all the on-shell correlation functions and in particular for the two-point function correlator $\langle F^a_{\mu\nu}F^{\mu\nu}_a\rangle$ (and $\langle R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\rangle$) on the torus, which is associated to the one-loop string corrections on the gauge coupling constant.

3 Non-zero $\mathbf{F}^{\mathbf{a}}_{\mu\nu}$ and $R^{\rho\sigma}_{\mu\nu}$ Background in Superstrings

Our aim is to define the deformation of the two-dimensional superconformal theory which corresponds to a non-zero field strength $F^a_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ background* and find the integrated one-loop partition function $Z^W(\mu, F, \mathcal{R})$, where F is by the magnitude of the field strength, $F^2 \equiv \langle F^a_{\mu\nu} F^{\mu\nu}_a \rangle$ and $\mathcal{R}$ is that of the curvature, $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle = \mathcal{R}^2$.

$$Z^W[\mu, F, \mathcal{R}] = \frac{1}{V(W)} \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} Z^W[\mu, F, \mathcal{R}; \tau, \bar{\tau}] \tag{3.1}$$

where $V(W)$ is the volume of the $W^{(4)}_k$ space; modulo the trivial infinity which corresponds to the one non-compact dimension, the remaining three-dimensional compact space is that of the three-dimensional sphere. In our normalization:

$$V(SU(2)_k) = \frac{1}{8\pi}(k+2)^{\frac{3}{2}}$$

so that it matches in the flat limit with the conventional flat space contribution.

In flat space, a small non-zero $F^a_{\mu\nu}$ background gives rise to an infinitesimal deformation of the 2-d σ-model action given by,

$$\Delta S^{2d}(F^{(4)}) = \int dz d\bar{z}\; F^a_{\mu\nu}[x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu] \bar{J}_a \tag{3.2}$$

Observe that for $F^a_{\mu\nu}$ constant (constant magnetic field), the left moving operator $[x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu]$ is not a well-defined $(1,0)$ operator on the world sheet. Even though the right moving Kac-Moody current $\bar{J}_a$ is a well-defined $(0,1)$ operator, the total deformation is not integrable in flat space. Indeed, the 2-d σ-model β-functions are not satisfied in the presence of a constant magnetic field. This follows from the fact that there is a *non-trivial back-reaction* on the gravitational background due the non-zero magnetic field.

The important property of $W^{(4)}_k$ space is that we can solve this back-reaction ambiguity. First observe that the deformation that corresponds to a constant magnetic field $B^a_i = \epsilon_{oijk} F^{ik}_a$ is a well-defined (1,1) integrable deformation, which breaks the $(2,2)$ superconformal invariance but preserves the $(1,0)$ world-sheet supersymmetry:

$$\Delta S^{2d}(W^{(4)}_k) = \int dz d\bar{z}\; B^a_i [I^i + \frac{1}{2}\epsilon^{ijk}\psi_j \psi_k] \bar{J}_a \tag{3.3}$$

where I^i is anyone of the $SU(2)_k$ currents. The deformed partition function is not zero due to the breaking of $(2,2)$ supersymmetry. In order to see that this is the correct replacement of the Lorentz current in the flat case, we will write the SU(2) group element as $g = \exp[i\vec{\sigma}\cdot\vec{x}/2]$ in which case $I^i = kTr[\sigma^i g^{-1}\partial g] = ik(\partial x^i + \epsilon^{ijk} x_j \partial x_k + \mathcal{O}(|x|^3))$. In the flat limit the first term corresponds to a constant gauge field and thus pure gauge so the only relevant term is the second one which corresponds to constant magnetic field in flat space. The $\mathcal{R}$ perturbation is

$$\Delta S(\mathcal{R}) = \int dz d\bar{z}\; \mathcal{R} \left[I^3 + \psi^1 \psi^2 \right] \bar{I}^3 \tag{3.4}$$

*Magnetic backgrounds in closed string theory have been also discussed in [28, 29, 30].

In σ-model language, in the flat limit it gives the following metric perturbation

$$\delta(ds^2) = -\mathcal{R}\left[x^1dx^2 - x^2dx^1\right]^2 \tag{3.5}$$

with constant Riemann tensor and scalar curvature equal to $6\mathcal{R}$. There is also a non-zero antisymmetric tensor with $H_{123} = 2\sqrt{\mathcal{R}}$ and dilaton $\delta\Phi = \mathcal{R}\left[(x^1)^2 + (x^2)^2 + 4(x^3)^2\right]/4$.

Due to the rotation invariance in S^3 we can choose $B_i^a = F\delta_i^3$ without loss of generality. The vector B_i^a indicates the direction in the gauge group space of the right-moving affine currents. Looking at the σ-model representation of this perturbation, we can observe that the $F_{\mu\nu}$ of this background gauge field is a monopole-like gauge field on S^3 and its lift to the tangent space is constant. Thus at the flat limit of the sphere it goes to the constant $F_{\mu\nu}$ background of flat space.

The moduli space of the F deformation is then given by the $SO(1,n)/SO(n)$ Lorentzian-lattice boosts with n being the rank of the right-moving gauge group. We therefore conclude that the desired partition function $Z^W(\mu, F, \mathcal{R}=0)$ is given in terms of the moduli of the $\Gamma(1,n)$ lorentzian lattice. The constant gravitational background $R^{ij}_{kl} = \mathcal{R}\epsilon^{3ij}\epsilon_{3kl}$ can also be included exactly by an extra boost, in which case the lattice becomes $\Gamma(1,n+1)$.

Let us denote by $\mathcal{Q}$ the fermionic lattice momenta associated to the left-moving $U(1)$ current $\partial H = \psi^1\psi^2$, by I the charge lattice of the left-moving $U(1)$ current associated to the I_3 current of $SU(2)_k$, by $\bar{\mathcal{Q}}$ the charge lattice of a right U(1) which is part of the Cartan algebra of the non-abelian right gauge group and by $\bar{I}$ the charge lattice of the right-moving $U(1)$ current associated to the $\bar{I}_3$ current of $SU(2)_k$. In terms of these charges the undeformed partition function can be written as

$$Tr[\exp[-2\pi\mathrm{Im}\tau(\mathrm{L}_0 + \bar{\mathrm{L}}_0) + 2\pi\mathrm{i}\mathrm{Re}\tau(\mathrm{L}_0 - \bar{\mathrm{L}}_0)]] \tag{3.6}$$

where

$$\mathrm{L}_0 = \frac{1}{2}\mathcal{Q}^2 + \frac{I^2}{k} + \cdots \;\; , \;\; \bar{\mathrm{L}}_0 = \frac{1}{2}\bar{\mathcal{Q}}^2 + \frac{\bar{I}^2}{k} + \cdots \tag{3.7}$$

where the dots stand for operators that do not involve $I, \bar{I}, \mathcal{Q}, \bar{\mathcal{Q}}$.

The (1,1) perturbation that turns on a constant gauge field strength F as well as a constant curvature $\mathcal{R}$ background produces an O(1,2) 2-parameter boost in $O(2,2)$, acting on the charge lattice above, which transforms L_0 and $\bar{\mathrm{L}}_0$ to

$$\mathrm{L}_0' = \mathrm{L}_0 + \frac{\cosh\psi - 1}{2}\left(\frac{(\mathcal{Q}+I)^2}{k+2} + \left(\cos\theta\frac{\bar{I}}{\sqrt{k}} + \sin\theta\frac{\bar{\mathcal{Q}}}{\sqrt{2}}\right)^2\right) + \tag{3.8}$$
$$+\sinh\psi\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}\left(\cos\theta\frac{\bar{I}}{\sqrt{k}} + \sin\theta\frac{\bar{\mathcal{Q}}}{\sqrt{2}}\right)$$

and

$$\mathrm{L}_0' - \bar{\mathrm{L}}_0' = \mathrm{L}_0 - \bar{\mathrm{L}}_0 \tag{3.9}$$

The parameters θ and ψ are related to the constant background fields F and $\mathcal{R}$ by†

$$F = \frac{\sinh\psi\sin\theta}{\sqrt{2(k+2)}} \;\; , \;\; \mathcal{R} = \frac{\sinh\psi\cos\theta}{\sqrt{k(k+2)}} \tag{3.10}$$

†The k-dependence is such that there is smooth flat space limit.

so that

$$\mathrm{L}_0' - \mathrm{L}_0 = (\mathcal{Q}+I)\left(\mathcal{R}\bar{I} + F\bar{\mathcal{Q}}\right) + \tag{3.11}$$

$$+\frac{\sqrt{1+(k+2)(2F^2+k\mathcal{R}^2)}-1}{2}\left(\frac{(\mathcal{Q}+I)^2}{k+2} + \frac{\left(\mathcal{R}\bar{I} + +F\bar{\mathcal{Q}}\right)^2}{(2F^2+k\mathcal{R}^2)}\right)$$

The first term is the standard perturbation while the second term is the back-reaction necessary for conformal and modular invariance. Expanding the partition function in a power series in $F, \mathcal{R}$

$$Z^W(\mu, F, \mathcal{R}) = \sum_{n,m=0}^{\infty} F^n \mathcal{R}^m Z^W_{n,m}(\mu) \tag{3.12}$$

we can extract the integrated correlators $\langle F^n R^m\rangle = Z_{n,m}$. For $\langle R\rangle$, $\langle F^2\rangle$, $\langle FR\rangle$ and $\langle R^2\rangle$ we obtain:

$$Z^W_{0,1}(\mu) = -4\pi \mathrm{Im}\tau \langle (\mathcal{Q}+I)\bar{I}\rangle \tag{3.13}$$

$$Z^W_{2,0} = 8\pi^2 \mathrm{Im}\tau^2 \left[\langle (\mathcal{Q}+I)^2\rangle\langle(\bar{\mathcal{Q}})^2\rangle - \frac{2\langle(\mathcal{Q}+I)^2\rangle + (k+2)\langle\bar{\mathcal{Q}}^2\rangle}{8\pi\mathrm{Im}\tau}\right] \tag{3.14}$$

$$Z^W_{1,1}(\mu) = 16\pi^2 \mathrm{Im}\tau^2 \langle \bar{I}\bar{\mathcal{Q}}\rangle \left[\langle (\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi\mathrm{Im}\tau}\right] \tag{3.15}$$

$$Z^W_{0,2}(\mu) = 8\pi^2 \mathrm{Im}\tau^2 \left[\langle (\mathcal{Q}+I)^2\rangle\langle\bar{I}^2\rangle - \frac{k\langle(\mathcal{Q}+I)^2\rangle + (k+2)\langle\bar{I}^2\rangle}{8\pi\mathrm{Im}\tau}\right] \tag{3.16}$$

The charges $\mathcal{Q}, \bar{\mathcal{Q}}$ in the above formulae act in the respective $\vartheta\left[{\alpha \atop \beta}\right](\tau, v)$-functions as differentiation with respect to v. In particular $\mathcal{Q}$ acts in the $\vartheta[{\alpha \atop \beta}]$ of eqs. (2.5), (2.9), $I, \bar{I}$ act in the level-k ϑ-function present in $\Gamma(SU(2)_k)$ (due to the parafermionic decomposition), and $\bar{\mathcal{Q}}$ acts on one of the right $\bar{\vartheta}$-functions.

It is straitforward to generalize the formulae above to the case where there are several gauge groups. These are generated by a collection of antiholomorphic currents $\bar{J}^i$ generating simple or $U(1)$ current algebras. We normalize them so that $\langle \bar{J}^i(z)\bar{J}^j(0)\rangle = k_i\delta^{ij}/2z^2$. This fixes the normalization of the quadratic Casimirs in the simple factors. Then,

$$\delta L_0 = \delta\bar{L}_0 = (\mathcal{Q}+I)(\mathcal{R}\bar{I} + F_i\bar{J}^i) + \tag{3.17}$$

$$+\frac{-1+\sqrt{1+(k+2)(k_iF_i^2+k\mathcal{R}^2)}}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + \frac{(F_i\bar{J}^i + \mathcal{R}\bar{I})^2}{k_iF_i^2+k\mathcal{R}^2}\right]$$

Expanding again in the background fields up to second order we obtain

$$\langle F_i\rangle = -4\pi\mathrm{Im}\tau\langle(\mathcal{Q}+I)\rangle\langle\bar{J}^i\rangle \tag{3.18a}$$

$$\langle \mathcal{R}\rangle = -4\pi\mathrm{Im}\tau\langle(\mathcal{Q}+I)\rangle\langle\bar{I}\rangle \tag{3.18b}$$

$$\langle F_i^2\rangle = 8\pi^2\mathrm{Im}\tau^2\left[\langle(\mathcal{Q}+I)^2\rangle\langle(\bar{J}^i)^2\rangle - \frac{k_i\langle(\mathcal{Q}+I)^2\rangle + (k+2)\langle(\bar{J}^i)^2\rangle}{8\pi\mathrm{Im}\tau}\right] \tag{3.18c}$$

$$\langle \mathcal{R}^2\rangle = 8\pi^2\mathrm{Im}\tau^2\left[\langle(\mathcal{Q}+I)^2\rangle\langle\bar{I}^2\rangle - \frac{k\langle(\mathcal{Q}+I)^2\rangle + (k+2)\langle\bar{I}^2\rangle}{8\pi\mathrm{Im}\tau}\right] \tag{3.18d}$$

$$\langle \mathcal{R}F_i\rangle = 16\pi^2\mathrm{Im}\tau^2\langle \bar{I}\bar{J}^i\rangle\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi\mathrm{Im}\tau}\right] \tag{3.18e}$$

$$\langle F_iF_j\rangle = 16\pi^2\mathrm{Im}\tau^2\langle \bar{J}^i\bar{J}^j\rangle\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi\mathrm{Im}\tau}\right] \tag{3.18f}$$

where we should rermember that $k+2 = 1/\mu^2$.

Renormalizations of higher terms can be easily computed. We give here the expression for an F_i^4 term,

$$\langle F_i^4\rangle = \frac{(4\pi\mathrm{Im}\tau)^4}{24}\langle\Big[(\mathcal{Q}+I)^4(\bar{J}^i)^4 - \frac{3}{4\pi\mathrm{Im}\tau}(\mathcal{Q}+I)^2(\bar{J}^i)^2(k_i(\mathcal{Q}+I)^2+(k+2)(\bar{J}^i)^2)+$$
$$+\frac{3}{4(4\pi\mathrm{Im}\tau)^2}[k_i(\mathcal{Q}+I)^2+(k+2)(\bar{J}^i)^2]^2 - \frac{3k_i(k+2)}{2(4\pi\mathrm{Im}\tau)^3}[k_i(\mathcal{Q}+I)^2+(k+2)(\bar{J}^i)^2]\Big]\rangle \tag{3.19}$$

4 One-loop Corrections to the Coupling Constants

The term linear in R provides us with the one loop renormalization of Newton's constant. It is obvious from (3.13) that this renormalization is zero to one-loop since the only term that might contribute from the left is $\mathcal{Q}$ and $\langle\bar{I}\rangle = 0$ due to global $SU(2)$ symmetry. Strictly speaking, what we have computed is the renormalization of a linear combination of Newton's constant and the axion-dilaton kinetic term. However we can disentangle the two by turning on a general $C_{ij}(\mathcal{Q}_i+I_i)\bar{I}_j$ background. For general constant C_{ij} this satisfies the string equations to leading order, and this is sufficient for computing the first order expectation value (3.13) relevant for the renormalization of Newton's constant. For this more general perturbation we still have

$$C_{ij}\langle(\mathcal{Q}_i+I_i)\bar{I}_j\rangle = 0 \tag{4.1}$$

again due to the global $SU(2)$ symmetry. The above implies that in all 4-D heterotic string models with flat spacetime, Newton's constant and the kinetic axion-dilaton terms do not renormalize at one-loop. This is true also for models where supersymmetry is spontaneously broken at tree level provided the one-loop cosmological constant is finite (no tachyons).

This argument generalizes in an obvious way to higher loops due to the vanishing of $\langle\bar{I}^3\rangle$ on any genus Riemann surface. This implies that Newton's constant and the axion, dilaton kinetic terms are not renormalized in perturbation theory for heterotic backgrounds with $N\geq 1$ spacetime supersymmetry.* An amusing fact is that Newton's constant does get a finite one-loop renormalization in the respective type-II backgrounds (from N=(1,1) sectors†). There we have

$$Z_{0,1}^{\mathrm{type-II}} = -2\pi\mathrm{Im}\tau\langle(\mathcal{Q}+I)(\bar{\mathcal{Q}}+\bar{I})\rangle \tag{4.2}$$

*We were informed by Minahan and Nemeschansky that they reached a similar conclusion at one loop.

†If we have higher spacetime supersymmetry the renormalization is zero due to extra zero modes.

Only the $\mathcal{Q}\bar{\mathcal{Q}}$ term saturates the 2d fermionic zero modes in this case and we obtain the following finite result

$$\langle R\rangle_{\text{one-loop}} \sim \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau^2}\frac{\Gamma(SU(2))}{V(SU(2))} = \frac{\pi}{3}\left(1+2\frac{\mu^2}{M_{str}^2}\right) \tag{4.3}$$

We have defined M_{str}^2 to be the mass of the lowest lying oscillator state in the string spectrum ($M_{str}^2 = 1/\alpha'$).

We now focus on the one-loop correction to the gauge couplings[‡], which is proportional to $Z_{2,0}^W(\mu)$. We can use the Riemann identity to transform the sum over the (α,β) ϑ-function characteristics (with non-zero v) that appear in (2.5), (2.9)

$$\frac{1}{2}\sum_{a,b=0}^{1}(-)^{\alpha+\beta+\alpha\beta}\vartheta[^{\alpha}_{\beta}](v|\tau)\vartheta[^{\alpha+h_1}_{\beta+g_1}](0|\tau)\vartheta[^{\alpha+h_2}_{\beta+g_2}](0|\tau)\vartheta[^{\alpha-h_1-h_2}_{\beta-g_1-g_2}](0|\tau) = \tag{4.4}$$

$$= \vartheta[^1_1](v/2|\tau)\vartheta[^{1-h_1}_{1-g_1}](v/2|\tau)\vartheta[^{1-h_2}_{1-g_2}](v/2|\tau)\vartheta[^{1+h_1+h_2}_{1+g_1+g_2}](v/2|\tau)$$

In this representation the charge operators are derivatives with respect to v.

We will focus for simplicity to heterotic $Z_2\times Z_2$ orbifolds. In this case all the characteristics in eq. (2.9) take the values $0,1$. The only non-zero contribution appears when one of the pairs (h_i,g_i) of twists is $(0,0)$ and the rest non-zero. There are three sectors where two out of the four fermion ϑ-functions depend only on $v/2$; they give non-zero contribution only when both derivatives with respect to v act on them. We have in total three $N=2$ sectors; the $N=4$ and the $N=1$ sectors give zero contribution in $Z_{2,0}(\mu)$ for the $Z_2\times Z_2$ orbifold model. For other orbifold models there might exist non-zero contributions coming from the N=1 sectors. Using the fact that the contribution to the partition function of the twisted bosons cancels (up to a constant that is proportional to the number of fixed points) that of the twisted fermions, and also the identity $\vartheta'(0)/2\pi = \eta^3$, we obtain the following formula for $Z_{2,0}(\mu)$:

$$Z_{2,0}^A(\mu) = -\sum_{i=1}^{3}\int_{\mathcal{F}}\frac{d\tau d\bar\tau}{\text{Im}\tau}\frac{\Gamma(SU(2))}{V(SU(2))}\frac{\Gamma_{2,2}^i(T_i,U_i)}{\bar\eta^{24}}\left[\bar{\mathcal{Q}}_A^2 - \frac{1}{4\pi\text{Im}\tau}\right]2\bar\Omega(\bar\tau) \tag{4.5}$$

where A indicates the appropriate gauge group (E_8, E_6 or $U(1)$), $\bar{\mathcal{Q}}_A$ is the associated charge operator, normalized so that it acts as $\frac{i}{\pi}\frac{\partial}{\partial\bar\tau}$ on the ϑ-functions and $\bar\Omega = \bar\Omega_8\bar\Omega_6$ with

$$\bar\Omega_8(\bar\tau) = \frac{1}{2}\sum_{a,b=0}^{1}\bar\vartheta^8[^a_b] \ \ , \ \ \bar\Omega_6(\bar\tau) = \frac{1}{4}\left[\bar\vartheta_2^8(\bar\vartheta_3^4+\bar\vartheta_4^4) - \bar\vartheta_4^8(\bar\vartheta_3^4+\bar\vartheta_2^4) + \bar\vartheta_3^8(\bar\vartheta_2^4-\bar\vartheta_4^4)\right] \tag{4.6}$$

Thus the one-loop corrected gauge coupling constant can be written as

$$\frac{16\pi^2}{g_A^2(\mu)} = \frac{16\pi^2}{g_A^2(M_{str})} + Z_{2,0}^A(\mu) \tag{4.7}$$

Eq. (4.5) applies to any 4-d symmetric orbifold string model, the only things that change are the moduli contribution from $\Gamma^i(T,U)$ and the specific form of $\bar\Omega$. This formula differs

[‡]Calculations similar in spirit for "topological" quantites have been done in [31].

from that of [9, 10, 11] since it includes the so-called universal contribution. In particular the back-reaction of gravity is included exactly and contributes to the universal terms. Taking differences between different gauge groups we obtain the regularized form of the result of [9, 10, 11]. This result will be also presented below. The only difference from their formula is the replacement of the flat space contribution by $\Gamma(SU(2))/V(SU(2))$. Our result is *explicitly modular invariant* and *finite* both in the IR and UV.

Before we proceed further some general remarks are in order. First it is obvious from (3.18) that N=4 sectors (having 4 zero modes) do not contribute to the renormalization of coupling constants. Each operator $\mathcal{Q}$ soaks up one zero mode. Since we have at most $\mathcal{Q}^2$ in our expressions N=4 sectors give a vanishing result. N=2 sectors have 2 zero modes so only terms that contain $\mathcal{Q}^2$ contribute. In such a case formulae (3.18) simplify to

$$\langle F_i \rangle_{N=2} = \langle \mathcal{R} \rangle_{N=2} = 0 \tag{4.8a}$$

$$\langle F_i^2 \rangle_{N=2} = 8\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2 \rangle \left[\langle (\bar{J}^i)^2 \rangle - \frac{k_i}{8\pi \mathrm{Im}\tau} \right] \tag{4.8b}$$

$$\langle \mathcal{R}^2 \rangle_{N=2} = 8\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2 \rangle \left[\langle \bar{I}^2 \rangle - \frac{k}{8\pi \mathrm{Im}\tau} \right] \tag{4.8c}$$

$$\langle \mathcal{R} F_i \rangle_{N=2} = 16\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2 \rangle \langle \bar{I} \bar{J}^i \rangle \tag{4.8d}$$

$$\langle F_i F_j \rangle_{N=2} = 16\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2 \rangle \langle \bar{J}^i \bar{J}^j \rangle \tag{4.8e}$$

Formula (4.8b) has been derived previously [11]. These formulae are also valid for N=1 sectors since the terms linear in $\mathcal{Q}$ that could contribute come with I_3 whose expectation value is zero.

In order to clearly see how the $W_k^{(4)}$ acts as an IR regulator, it is convenient to perform the summation on the spin index l of the $SU(2)$ characters. This sum can be done analytically and one obtains the following surprising (and eventually useful) identity

$$\Gamma(SU(2)_k) = \sqrt{\mathrm{Im}\tau} \frac{(k+2)^{3/2}}{8\pi} \left[\frac{\partial Z(R)}{\partial R} |_{R^2=k+2} - \frac{1}{2} (R \to R/2) \right] \tag{4.9}$$

where $Z(R)$ is the $\Gamma(1,1)$ lattice contribution of the torus:

$$Z(R) = \sum_{m,n} \exp \left[\frac{i\pi\tau}{2} \left(\frac{m}{R} + nR \right)^2 - \frac{i\pi\bar{\tau}}{2} \left(\frac{m}{R} - nR \right)^2 \right] \tag{4.10}$$

Notice that the derivative with respect to R in (4.9) subtracts the $(m,n) = (0,0)$ contribution which is responsible for the IR divergence. In particular we have that the infrared cutoff $\mu = 1/R$.

Using (4.9), $Z_2(\mu)$ becomes,

$$Z_2^A(\mu) = \sum_{i=1}^{3} 2 \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau^2} \mathrm{Im}\tau^{\frac{1}{2}} \left[Z'(R)|_{k+2} - \frac{1}{2} (R \to R/2) \right] \left[\mathrm{Im}\tau \Gamma^i_{(2,2)}(T_i, U_i) \Sigma^A \right] \tag{4.11}$$

The function Σ^A depends on the gauge group in question and its constant part is proportional to the β-function contribution of the N=2 sectors, $b_i = C(g_A) - T(R^i_A)$. For example, in the E_8 case it is given by

$$\Sigma^{E_8} = -2\frac{\bar{\Omega}_6}{\bar{\eta}^{24}}\left[\frac{i\partial}{\pi\partial\bar{\tau}} - \frac{2}{\pi \mathrm{Im}\tau}\right]\bar{\Omega}_8 \tag{4.12}$$

The differential operator in (4.12) acts as a covariant derivative on modular forms.

Eq. (4.11) is the final form for the complete string one-loop radiative correction to the appropriate gauge couplings. This result is finite and manifestly invariant under the target space duality group that acts on the T_i, U_i moduli. We see in particular that the (regulated) integrand in our case is related to the partition function of a (3,3) lattice at special values of the (3,3) moduli. The derivative with respect to the R modulus is responsible for the regulation of the IR. In order to see this we will evaluate the part of the radiative correction coming from the low-lying states (massless in the limit $\mu \to 0$), which in the unregulated case is responsible for the IR divergence. This is achieved by replacing the (2,2) lattice contribution in eq. (4.11) by 1 and leaving apart for the moment the universal contribution (which is IR finite):

$$Z_2^{\mathrm{massless}}(\mu) = \left[\sum_{i=1}^{3} b_i\right]\int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau}\mathrm{Im}\tau^{\frac{1}{2}}\left[Z'(R)|_{k+2} - \frac{1}{2}\left(R \to R/2\right)\right] \tag{4.13}$$

As expected, $Z_2^{\mathrm{massless}}(\mu)$ turns out to be finite and for small μ behaves like

$$Z_2^{m=\mu}(\mu) = (b_1 + b_2 + b_3)[\log(M^2_{str}/2\mu^2) + 2c_0] + ... \tag{4.14}$$

where the dots stand for terms vanishing in the limit $\mu \to 0$. The constant c_0 can be computed exactly with the result

$$c_0 = \frac{3}{2} - \frac{1}{2}\log(\pi/2) + \frac{1}{2}\gamma_E - \frac{3}{4}\log(3) = 0.738857... \tag{4.15}$$

Observe that in this example the N=2 β-function coefficients add to the full N=1 β-function coefficient, $b_1 + b_2 + b_3 = 3C(g_a) - T(R_a)$. This is always the case in any string ground state where the Green-Schwarz duality anomaly cancellation is unnecessary [10, 11]. The constant coefficient c_0, together with that of massive states $F(T_i, U_i)$ as well as the universal contribution define unambiguously the string scheme and can thus be compared with the field theory result (regularized in the IR in the same way as above) in any UV scheme, for instance the conventional $\bar{DR}$. Although this coefficient is small, one has to compute the parts left over including the moduli dependence. In particular the universal contribution can be important. We calculate here the universal contribution due to would be massless states, e.g. the constant part of $\bar{\Omega}/\bar{\eta}^{24}$. This is equal to

$$\frac{60}{\pi}\int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2}\sqrt{\mathrm{Im}\tau}\left[Z'(R)|_{k+2} - \frac{1}{2}\left(R \to R/2\right)\right] = 20 + \mathcal{O}(\mu) \tag{4.16}$$

This contributes to the coefficient c_0 in (4.14) equal to $1/3$ for E_8 and $-5/21$ for E_6. It implies that a full calculation is necessary, namely the contributions from all massive states, in order to find the exact string scheme.

We will now evaluate the integrals (4.11) over the torus moduli space in order to obtain the full one-loop corrections to the coupling constants.

As seen previously the one loop corrections involve integrals of the form:

$$I(R,U,T)=\frac{\partial}{\partial R}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}[\sqrt{\mathrm{Im}\tau}Z(R)][\mathrm{Im}\tau\Gamma_{2,2}(U,T)]\left(\bar{F}_1+\frac{1}{\mathrm{Im}\tau}\bar{F}_2\right) \tag{4.17}$$

Here $\bar{F}_1$, $\bar{F}_2$ are antiholomorphic functions with the following expansions

$$\bar{F}_1=\sum_{m=m_0}^{\infty}C_m e^{-2\pi i m\bar{\tau}}$$

$$\bar{F}_2=\sum_{m=\hat{m}_0}^{\infty}\hat{C}_m e^{-2\pi i m\bar{\tau}}$$

where $m_0=0$ in the case of gauge coupling constant corrections and $m_0=-1$ for the $\mathcal{R}^2$ coupling constant correction. $\hat{m}_0$ is -1 in both cases. Moreover $\bar{F}_1+\bar{F}_2/\mathrm{Im}\tau$ is modular invariant.

For example, for the case of the E_8 correction we have

$$Z^{E_8}_{2,0}(\mu)=\sum_{i=1}^{3}[I(1/\mu,U_i,T_i)-I(1/2\mu,U_i,T_i)] \tag{4.18}$$

with

$$\bar{F}_1=-\frac{2i}{\pi}\frac{\bar{\Omega}_6}{\bar{\eta}^{24}}\partial_{\bar{\tau}}\bar{\Omega}_8\quad,\quad\bar{F}_2=\frac{4}{\pi}\frac{\bar{\Omega}_6\bar{\Omega}_8}{\bar{\eta}^{24}} \tag{4.19}$$

The integral (4.17) can be evaluated using the method of orbits of the modular group in order to unfold the integration domain. There are three contributions, that of the zero orbit I_0, the non-degenerate orbits I_1 and the degenerate orbits I_2 The perturbative IR divergence exists in I_2 although there are extra divergences at special points in target moduli space (e.g. $T=U$, $T=U=i$ and $T=U=\rho$).

We obtain

$$\begin{aligned}\frac{I_0}{\mathrm{Im}T}&=\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}\left(\bar{F}_1+\bar{F}_2/\mathrm{Im}\tau\right)+\mathcal{O}(\mu^2)=\\&=\left[\frac{2}{\pi}\bar{G}_2\bar{F}_1+\frac{2}{\pi^2}\bar{G}_2^2\bar{F}_2\right]_{\bar{q}^0\mathrm{term}}+\mathcal{O}(\mu^2)=\\&=-\frac{2\pi}{3}(C_0-24C_{-1})+\frac{2\pi^2}{9}(\hat{C}_0-48\hat{C}_{-1})+\mathcal{O}(\mu^2)\end{aligned} \tag{4.20}$$

$$\begin{aligned}I_1=-2\sum_{k=1}^{\infty}\sum_{l=[m_0/k]}^{\infty}&C_{kl}\left[\log[1-e^{-2\pi i(k\bar{T}-lU)}]+c.c.\right]-\\&-2\sum_{k=1}^{\infty}\sum_{l=[\hat{m}_0/k]}^{\infty}\hat{C}_{kl}\left[\log[1-e^{-2\pi i(k\bar{T}-lU)}]+\right.\\&\left.+\frac{1}{4\pi\mathrm{Im}(kT+lU)}F(e^{-2\pi i(k\bar{T}-lU)})+c.c.\right]+\mathcal{O}(\mu^2)\end{aligned} \tag{4.21}$$

with

$$\bar{G}_2 = \frac{\pi^2}{3}\left[1 - 24\sum_{n=1}^{\infty}\frac{n\bar{q}^n}{1-\bar{q}^n}\right], \tag{4.22}$$

$F(x)$ is related to the dilogarithm function

$$F(x) = \int_0^x \frac{du}{u}\log(1-u) \tag{4.23}$$

and $[x]$ stands for the integer part of x.

Finally

$$I_3 = \frac{\mathrm{Im}U\mathrm{Im}T}{\pi}\left(1-\mu\frac{\partial}{\partial\mu}\right)\sum_{m,n}{}'\sum_{j,p}{}'\frac{\mu^2 e^{-2\pi imn/\mu^2}}{\mathrm{Im}T\,\mu^2\,|j+Up|^2+\mathrm{Im}U\,m^2}\times \tag{4.24}$$

$$\times\left[C_{mn}+\frac{\mathrm{Im}U}{\pi}\frac{\mu^2\hat{C}_{mn}}{\mathrm{Im}T\,\mu^2\,|j+Up|^2+\mathrm{Im}U\,m^2}\right]$$

The above results imply that for differences of gauge couplings[§]the regulated result is proportional to

$$\Delta_{AB}\sim -4\mathrm{Re}\log\eta(T)+2\left[\pi\mathrm{Im}T\mu^2\sum_{j,p}{}'\sinh^{-2}\left(\pi\mu\sqrt{\frac{\mathrm{Im}T}{\mathrm{Im}U}}|j+Up|\right)-\frac{1}{2}\left(\mu\to 2\mu\right)\right]+\mathcal{O}(\mu^2) \tag{4.25}$$

We can expose the IR divergent part as well as the duality invariance of the result above by approximating for small x

$$\frac{1}{\sinh^2 x}\approx\frac{1}{x^2(1+x^2/6)^2}\approx\frac{1}{x^2}-\frac{1}{x^2+3} \tag{4.26}$$

Then, we obtain

$$\Delta_{AB}\sim\log\frac{M_{str}^2}{\mu^2}-\log\left[|\eta(T)\eta(U)|^4T_2U_2\right]+\log\frac{3e^{2\gamma_E}}{\pi^2}+\mathcal{O}(1) \tag{4.27}$$

where the $\mathcal{O}(1)$ piece is moduli independent. A more careful control of the subleading terms is needed in order to compute this constant part The moduli dependent part agrees with the result of ref. [9].

Although the general formulae (4.20)-(4.24) are rather explicit, it will be useful to cast them in a form where the T, U target space duality is manifest.

Before ending this section we give also the regularized one-loop correction to the $\mathcal{R}^2$ coupling:

$$Z^{\mathrm{grav}}(\mu) = -\sum_i\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}\mathrm{Im}\tau\Gamma_{2,2}(T_i,U_i)\left[\bar{I}^2-\frac{1-2\mu^2}{8\pi\mu^2\mathrm{Im}\tau}\right]\frac{\Gamma_{SU(2)}(\mu)}{V_{SU(2)}(\mu)}\frac{2\bar{\Omega}}{\bar{\eta}^{24}}$$

[§]We exclude here $U(1)$'s that can get enhanced at special points.

$$= \sum_i \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2} \mathrm{Im}\tau \Gamma_{2,2}(T_i, U_i) \frac{2\bar{\Omega}}{\bar{\eta}^{24}} X(\mu) \tag{4.28}$$

where

$$X(\mu) = \sum_{m,n \in Z} (-1)^{m+n+mn} \left[\left(4\pi i \partial_{\bar{\tau}} \log \bar{\eta} - \frac{\pi}{\mathrm{Im}\tau} \right) \left(1 - \frac{2\pi |m - n\tau|^2}{4\mu^2 \mathrm{Im}\tau} \right) + \right. \tag{4.29}$$

$$\left. + \frac{4\pi^2 (m - n\tau)^2}{48\mu^4 \mathrm{Im}\tau^2} \left(3 - \frac{2\pi |m - n\tau|^2}{4\mu^2 \mathrm{Im}\tau} \right) \right] \exp \left[- \frac{\pi |m - n\tau|^2}{4\mu^2 \mathrm{Im}\tau} \right]$$

In general, the coupling constant corrections depend on several scales, namely the expectation values of the moduli (here T_i and U_i) and the infrared scale μ. In the next section we will show that the one-loop corrections to the couplings obey differential equations which relate changes of the various scales above.

5 IR Flow Equations for Couplings

Once we have obtained the one-loop corrections to the coupling constants we can observe that they satisfy scaling type flows. We will present here IR Flow Equations (IRFE) for differences of gauge couplings.

The existence of IRFE is due to differential equations satisfied by the lattice sum of an arbitrary (d,d) lattice,

$$Z_{d,d} = \mathrm{Im}\tau^{d/2} \sum_{P_L, P_R} e^{i\pi\tau P_L^2/2 - i\pi\bar{\tau} P_R^2/2} \tag{5.1}$$

where

$$P_{L,R}^2 = \vec{n} G^{-1} \vec{n} + 2\vec{m} B G^{-1} \vec{n} + \vec{m}[G - BG^{-1}B]\vec{m} \pm 2\vec{m} \cdot \vec{n} \tag{5.2}$$

$\vec{m}, \vec{n}$ are integer d-dimesional vectors and G_{ij} (B_{ij}) is a real symmetric (antisymmetric) matrix. $Z_{d,d}$ is $O(d, d, Z)$ and modular invariant. Moreover it satisfies the following second order differential equation*:

$$\left[\left(G_{ij} \frac{\partial}{\partial G_{ij}} + \frac{1-d}{2} \right)^2 + 2 G_{ik} G_{jl} \frac{\partial^2}{\partial B_{ij} \partial B_{kl}} - \frac{1}{4} - 4 \mathrm{Im}\tau^2 \frac{\partial^2}{\partial\tau \partial\bar{\tau}} \right] Z_{d,d} = 0 \tag{5.3}$$

The equation above involves also the modulus of the torus τ. Thus it can be used to convert the integrands for threshold corrections to differences of coupling constants into total derivatives on τ-moduli space. Using the equation above for the $\Gamma_{1,1}(\mu)$ and $\Gamma_{2,2}(T, U)$ lattices we can evaluate the τ integral and we are left with a differential equation for the couplings with respect to the T, U moduli and the IR scale μ only. To derive such an equation we start from eq. (4.11) to obtain

$$\Delta_{AB} \equiv \frac{16\pi^2}{g_A^2} - \frac{16\pi^2}{g_B^2} = (b_A - b_B) \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2} A(R) B(T, \bar{T}, U, \bar{U}) \tag{5.4}$$

*The special case for $d = 2$ of this equation was noted and used in [9, 11].

with

$$A(R) = 2\sqrt{\text{Im}\tau}\left[Z'(R)|_{k+2} - \frac{1}{2}\left(R \to R/2\right)\right] \tag{5.5}$$

$$B(T,\bar{T},U,\bar{U}) = \text{Im}\tau\Gamma_{2,2}(T,\bar{T},U,\bar{U}) \tag{5.6}$$

Eq. (5.4) does not apply to $U(1)$'s that can get enhanced at special points of the moduli. The general equation (5.3) translates to the following equations for A and B:

$$\frac{1}{16}\left[\left(\frac{\partial}{\partial R}R\right)^2 - 1\right]A(R) = \text{Im}\tau^2\partial_\tau\partial_{\bar{\tau}}A(R) \tag{5.7}$$

$$\text{Im}T^2\partial_T\partial_{\bar{T}}B(T,\bar{T},U,\bar{U}) = \text{Im}\tau^2\partial_\tau\partial_{\bar{\tau}}B(T,\bar{T},U,\bar{U}) \tag{5.8}$$

and a similar one with $T \to U$. Using (5.7), (5.8) we obtain

$$\left[\left(\frac{\partial}{\partial R}R\right)^2 - 1 - 16\text{Im}T^2\partial_T\partial_{\bar{T}}\right]\Delta_{AB} = 16(b_A - b_B)\int_{\mathcal{F}} d^2\tau\left[\partial_\tau\left(B\partial_{\bar{\tau}}A\right) - \partial_{\bar{\tau}}\left(A\partial_\tau B\right)\right] \tag{5.9}$$

The righthand side in (5.9) is a total divergence in moduli space, getting contributions only from $\tau \to i\infty$. However the contribution there is zero due to the IR cutoff (unlike the unregulated case). Thus, using $R = 1/\mu$, eq. (5.9) becomes

$$\left[\left(\mu\frac{\partial}{\partial\mu}\right)^2 - 2\mu\frac{\partial}{\partial\mu} - 16\text{Im}T^2\frac{\partial^2}{\partial T\partial\bar{T}}\right]\Delta_{AB} = 0 \tag{5.10}$$

and we have also a similar one with $T \to U$.

We strongly believe that such equations also exist for single coupling constants using appropriate differential equations for $(d, d+n)$ lattices.

Notice first that the IR scale μ plays the role of the RG scale in the effective field theory (see eqs. (4.7) and (4.14)):

$$\frac{16\pi^2}{g_A^2(\mu)} = \frac{16\pi^2}{g_A^2(M_{str})} + b_A\log\frac{M_{str}^2}{\mu^2} + F_A(T_i) + \mathcal{O}(\mu^2/M_{str}^2) \tag{5.11}$$

where the moduli T_i have been rescaled by M_{str} so they are dimensionless. Second, the IRFE gives a scaling relation for the moduli dependent corrections. Such relations are very useful for determining the moduli dependence of the threshold corrections. We will illustrate below such a determination, applicable to the $Z_2 \times Z_2$ example described above.

Using the expansion (5.11) and applying the IRFE (5.10) we obtain

$$\text{Im}T^2\frac{\partial^2}{\partial T\partial\bar{T}}(F_A - F_B) = \frac{1}{4}(b_A - b_B) \tag{5.12}$$

and a similar one for U. This non-homogeneous equation has been obtained in [9, 11].

Solving them we obtain

$$F_A - F_B = (b_B - b_A)\log[\text{Im}T\text{Im}U] + f(T,U) + g(T,\bar{U}) + \text{cc} \tag{5.13}$$

If at special points in moduli space, the extra massless states are uncharged with respect to the gauge groups appearing in (5.12) then the functions f and g are non-singular inside moduli space. In such a case duality invariance of the threshold corrections implies that

$$F_A - F_B = (b_B - b_A) \log[\mathrm{Im}T\mathrm{Im}U|\eta(T)\eta(U)|^4] + \mathrm{constant} \tag{5.14}$$

This is the result obtained via direct calculation in [9].

It is thus obvious that the IRFE provides a powerful tool in evaluating general threshold corrections as manifestly duality invariant functions of the moduli.

6 Further Directions

We have presented an IR regularization for string theory (and field theory) induced by the curvature of spacetime as well as by non-trivial dilaton and axion fields. This regularization preserves a form of spacetime supersymmetry and gives masses to all massless fields (including chiral fermions) that are proportional to the curvature. In particular, the theory is IR finite also at special values of the moduli with extra massless states*.

In the regulated string theory we can compute exactly the one-loop effective action for arbitrarily large, constant, non-abelian gauge and gravitational fields. Using this result, among other things, we can compute unambiguously the string-induced one-loop threshold corrections to the gauge couplings as functions of the moduli.

Another set of important couplings that we have not explicitly addressed in this paper are the Yukawa couplings. Physical Yukawa couplings depend on the kähler potential and the superpotential. The superpotential receives no perturbative contributions and thus can be calculated at tree level. The Kḧaler potential however does get renormalized so in order to compute the one-loop corrected Yukawa couplings we have to compute the one-loop renormalization of the Kähler metric. When the ground state has (spontaneously-broken) spacetime supersymmetry the wavefunction renormalization of the scalars ϕ_i are the same as those for their auxiliary fields F_i. Thus we need to turn on non-trivial F_i, calculate their effective action on the torus and pick the quadratic part proportional to $F_i\bar{F}_{\bar{j}}$. This can be easily done using the techniques we developed in this paper since it turns out that the vertex operators [33] for some relevant F fields are bilinears of left and right U(1) chiral currents. The explicit computations will presented elsewhere.

There are several open problems that need to be addressed in this context.

Although we have obtained an explicit formula for threshold corrections more work is needed so that it is cast in form where all the duality symmetries are manifest.

The structure of higher loop corrections should be investigated. A priori there is a potential problem, due to the dilaton, at higher loops. One would expect that since there is a region of spacetime where the string coupling become arbitrarily strong, higher order computations would be problematic. We think that this is not a problem in our models,

*See ref. [32] for a recent attempt to take into account such points in the unregulated approach.

because in Liouville models with N=4 superconformal symmetry (which is the case we consider) there should be no divergence due to the dilaton at higher loops. However, this point need further study. One should eventually analyze the validity of non-renormalization theorems at higher loops [11] since they are of prime importance for phenomenology.

Once the full one-loop coupling corrections are known, and in the absence of higher order (perturbative) moduli-dependent corrections, it might be possible to implement the S-duality conjecture [34, 35] in order to obtain non-perturbative results [36] concerning the effective field theory of string theory.

The consequences of string threshold corrections for low energy physics should be studied in order to be able to make quantitative predictions.

Finally, the response of string theory to the magnetic backgrounds studied in this paper should be analysed since it may provide with useful clues concerning the behavior of strings in strong background fields and/or singularities.

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